

k -FLAT STRUCTURES AND EXOTIC CHARACTERISTIC CLASSES

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ABSTRACT. We generalize the concept of “foliation” and define k -flat structures; these are smooth vector bundles with affine connections whose characteristic forms vanish above a certain dimension. Using semisimplicial techniques we construct a classifying space for k -flat structures, and prove a classification theorem for these structures on smooth manifolds.

Techniques from rational homotopy theory are used to relate the exotic characteristic classes of foliations to the rational homotopy groups and cohomology of the classifying space.

Introduction. A familiar technique in algebraic topology is to reduce the problem of studying structures on a large collection of objects to the study of a universal example which “contains” all the others. Consider the problem of classifying vector bundles over CW complexes, in this case we need only look at spaces BO^k and BU^k , and their canonical k -plane bundles [9]. A nice feature of this example is that the algebraic invariants of k -plane bundles (i.e. Stiefel-Whitney classes, Chern classes, Euler class) are contained in the cohomology of the classifying spaces.

In this paper, we look at a set of invariants which can be attached to foliated smooth manifolds. These are the so-called “exotic characteristic classes” of foliations which were constructed in the late 1960’s independently by Bott, Haefliger, Gelfand-Fuks, Kamber-Tondeur, and Malgrange; they can be defined using purely geometric techniques (i.e. see [1]).

At about the same time, Haefliger [5] developed a topological classification of foliations using $B\Gamma$ spaces, and has applied the theory to smooth manifolds via theorems of Thurston and Gromov-Phillips. One would like to demonstrate a relationship between the exotic characteristic classes and the cohomology of the classifying spaces as in the example above, although this has proven to be difficult. Nevertheless, a lot is known about the homotopy type of $B\Gamma_k$, largely through the efforts of Hutsch, Hurder, Kamber-Tondeur, Rasmussen, Thurston, et al.

In §1, we introduce k -flat structures, which generalize the concept of foliations. These are smooth vector bundles with affine connections whose characteristic forms vanish above dimension $2k$; they constitute the largest classes of objects for which exotic characteristic classes can be defined. k -flat structures and exotic characteristic classes were studied by Lehmann in [17].

Using semisimplicial techniques, we will construct classifying spaces for k -flat structures and prove a classification theorem for these structures on smooth manifolds. §2 contains a brief review of semisimplicial methods, while §3 contains the constructions and proofs.

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In §4, we use methods from rational homotopy theory to relate the exotic characteristic classes to the rational homotopy groups and cohomology of the classifying spaces. These methods were introduced by Hurder [6] to study the connection between the homotopy type of $B\Gamma_k$ and the exotic classes.

§5 contains examples and concluding remarks.

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1. Let M be a finite dimensional C^∞ manifold and let ζ be a C^∞ \mathbf{R}^k -vector bundle over M . $C^\infty(\zeta)$ will denote the space of smooth sections of ζ with the Whitney C^∞ topology, $C^\infty(M)$ is the space of smooth real valued functions on M , and T^*M is the tangent bundle to M . Recall that an *affine connection* on ζ is an \mathbf{R} -linear map $\nabla: C^\infty(T^*M) \times C^\infty(\zeta) \rightarrow C^\infty(\zeta)$ satisfying

1. $\nabla(fX, s) = f\nabla(X, s)$,
2. $\nabla(X, fs) = X(f)(s) + f\nabla(X, s)$

for $X \in C^\infty(T^*M)$, $s \in C^\infty(\zeta)$, $f \in C^\infty(M)$.

Associated to ∇ is its curvature tensor

$$K_\nabla(X, Y, s) = \nabla(X, \nabla(Y, s)) - \nabla(Y, \nabla(X, s)) - \nabla([X, Y], s).$$

We can describe ∇ and K_∇ locally in the following way: Choose a framed open cover $\{U_\alpha, s_1^\alpha, \dots, s_k^\alpha\}$ for M , and let $\omega^\alpha = (\omega_{ij}^\alpha)$ and $\Omega^\alpha = (\Omega_{ij}^\alpha)$ be the $k \times k$ matrices of 1- and 2-forms defined by

$$\nabla(X, s_i^\alpha) = \sum_j \omega_{ij}^\alpha(X) s_j^\alpha, \quad K_\nabla(X, Y, s_i^\alpha) = \sum_j \Omega_{ij}^\alpha(X, Y) s_j^\alpha.$$

∇ and K_∇ are completely determined by these local matrices, and ω^α and Ω^α are related by the identity $\Omega^\alpha = d\omega^\alpha - \omega^\alpha \wedge \omega^\alpha$.

All C^∞ bundles admit affine connections, and we can define the characteristic classes of a vector bundle using a connection and its curvature tensor via the Chern-Weil construction, which we briefly describe below.

Let $I(Gl_k)$ be the graded algebra of Gl_k -invariant polynomials $P: Gl_k \rightarrow \mathbf{R}$. $I_d(Gl_k)$ is the vector space of polynomials which are homogeneous of degree d , and [1] proves that $I(Gl_k) \cong [p_1, p_2, \dots, p_k]$, where p_i has degree i .

Suppose ∇ is an affine connection on ζ and $\{U_\alpha; s_1^\alpha, \dots, s_k^\alpha\}$ is a framed open cover for ∇ . Let $\Omega^\alpha, \omega^\alpha$ be as above. Then for any $P \in I_d(Gl_k)$

$$P(\Omega^\alpha)/U_\alpha \cap U_\beta = P(\Omega^\beta)/U_\alpha \cap U_\beta.$$

Since $K(fX, gY)(hs) = fghK(X, Y)(S)$ for $f, g, h \in C^\infty(M)$ [9], there is a $2d$ -form $P(K_\nabla)$ on M such that $P(K_\nabla) \mid U_\alpha = P(\Omega^\alpha)$.

Let $I = \mathbf{R}[c_1, c_2, \dots]$ be the free polynomial algebra on generators c_i in dimension $2i$ with trivial differential, and let Ω^*M denote the C^∞ de Rham algebra on M . Define a map $\nabla_*: I \rightarrow \Omega^*M$ by $\nabla_*c_i = p_i(K_\nabla) \in \Omega^{2i}M$ if $i \leq k$ and $\nabla_*c_i = 0$ if $i > k$. ∇_*c_i is the i th Chern polynomial of ∇ . Since $dp_i(K_\nabla) = 0$ for all i , ∇_* is a chain map and $[\nabla_*c_{2i}] \in H^{2i}(M, \mathbf{R})$ is the i th Pontrjagin class of ζ [9].

If ζ is equipped with certain types of extra structure, the Chern-Weil construction can be extended to give secondary invariants related to that structure. For example, if ζ is the normal bundle to a codimension k foliation \mathcal{F} , ζ admits an *adapted connection* $\nabla(\mathcal{F})$, which has the property that $K_{\nabla(\mathcal{F})}(X, Y, s) = 0$ whenever X and Y are vector fields tangent to \mathcal{F} [2]. It follows that $\nabla(\mathcal{F})_*$ is the zero map in dimensions greater than $2k$. Therefore, if I_k is the quotient of I by all elements of total degree $> 2k$, then $\nabla(\mathcal{F})_*$ induces a chain map from I_k to Ω^*M . Define a differential graded algebra $WO_k = I_k \otimes E_{\mathbf{R}}(h_1, h_3, \dots, h_e)$ where h_i has degree $2i - 1$, l is the largest odd integer less than or equal to k , and $d(c_i \otimes 1) = 0$, $d(1 \otimes h_i) = c_i \otimes 1$. It is proved in [1] that $\nabla(\mathcal{F})_*: I_k \rightarrow \Omega^*M$ extends to a chain map $\nabla(\mathcal{F})_*: WO_k \rightarrow \Omega^*M$ whose induced map in cohomology depends only on the concordance class of \mathcal{F} . The classes in the image $H(\nabla(\mathcal{F}))_*(H^*WO_k)$ are the *exotic characteristic classes* of \mathcal{F} .

If the normal bundle ζ of \mathcal{F} is a trivial bundle, the map $\nabla(\mathcal{F})_*$ extends to a larger complex: $W_k = I_k \otimes E_{\mathbf{R}}(h_1, h_2, \dots, h_k)$ such that $\text{degree}(h_i) = 2i - 1$ and $d(c_i \otimes 1) = 0$, $d(1 \otimes h_i) = c_i \otimes 1$. As above, the induced map $H\nabla(\mathcal{F})_*: H^*W_k \rightarrow H^*M$ depends only on the concordance class of \mathcal{F} .

We remark that in both the trivial and nontrivial cases, the definition of the characteristic classes relies only on the adapted connection $\nabla(\mathcal{F})$ and its vanishing property. This suggests the following generalization:

An affine connection ∇ on ζ is *k-flat* if $\nabla_*: I \rightarrow \Omega^*M$ is the zero map in dimensions greater than $2k$. (ζ, ∇) is a *k-flat structure* on M if ∇ is a *k-flat* connection on ζ . A map of *k-flat* structures $\phi: (\zeta, \nabla) \rightarrow (\zeta', \nabla')$ on M is a vector bundle isomorphism $\phi: \zeta \rightarrow \zeta'$ such that, if s and s' are sections of ζ and ζ' respectively so that

$$\begin{array}{ccc} \zeta & \xrightarrow{\phi} & \zeta' \\ s \downarrow & & \downarrow s' \\ M & \xrightarrow{\phi} & M \end{array}$$

commutes, then

$$\nabla(X, s)(m) = \nabla'(Y, s')(\phi(m))$$

where Y is a vector field such that $d\phi(X(m)) = Y(\phi(m))$. If (ζ, ∇) is a *k-flat* structure on M and ζ is a trivial bundle, a *trivialization* for (ζ, ∇) is a map $\phi: (M \times \mathbf{R}^k \rightarrow M, \nabla') \rightarrow (\zeta, \nabla)$ of *k-flat* structures.

Suppose $f: N \rightarrow M$ is a C^∞ map and (ζ, ∇) is a *k-flat* structure on M . We can define an affine connection $f^*\nabla$ on $f^*\zeta$ as follows: Choose a framed open cover $\{U_\alpha; s_1^\alpha, \dots, s_k^\alpha\}$ for ζ . Then $\{f^{-1}(U_\alpha), f^*s_1^\alpha, \dots, f^*s_k^\alpha\}$ is a framed open cover for $f^*\zeta$ and if $X \in C^\infty(T_*f^{-1}(U))$, $y \in f^{-1}(U)$, we define

$$f^*\nabla(X, f^*s_i^\alpha)(y) = (y, \nabla(X', s_i^\alpha))(f(y))$$

where $X' \in C^\infty(T^*M)$ satisfies $X'(f(y)) = df(X(y))$.

If $w^\alpha = (w_{ij}^\alpha)$ and $\Omega^\alpha = (\Omega_{ij}^\alpha)$, are the connection and curvature matrices of ∇ on $\{U_\alpha; s_1^\alpha, \dots, s_k^\alpha\}$, f^*w^α and $f^*\Omega^\alpha$ are the connection and curvature matrices of $f^*\nabla$ on

$$\{f^{-1}(U_\alpha), f^*s_1^\alpha, \dots, f^*s_k^\alpha\}.$$

Any relations among the Ω_{ij}^α 's must also be satisfied by $f^*\Omega_{ij}^\alpha$'s; in particular, ∇ *k-flat* implies $f^*\nabla$ is *k-flat*. Therefore, we have an induced *k-flat* structure $f^*(\zeta, \nabla) = (f^*\zeta, f^*\nabla)$ on N .

k -flat structures (ζ_0, ∇_0) and (ζ_1, ∇_1) on M are *concordant* if there is a k -flat structure (ζ, ∇) on $M \times I$ such that $i_j^*(\zeta, \nabla) = (\zeta_j, \nabla_j)$ for $j = 0, 1$. Let $F_k(M)$ denote the set of k -flat structures on M . Concordance is an equivalence relation on $F_k(M)$ and we let $cF_k(M) = F_k(M)/\sim$. If $f: N \rightarrow M$ is as above, the map $f^*: F_k(M) \rightarrow F_k(N)$ preserves concordance classes and induces a map $f^*: cF_k(M) \rightarrow cF_k(N)$. Note that if $(\zeta, \nabla) \sim (\zeta', \nabla') \in F_k(M)$, then ζ is isomorphic to ζ' . Conversely, given a particular vector bundle ζ , we will want to consider the set of concordance classes $\{[(\zeta'_1, \nabla')]\in cF_k(M) \mid \zeta' \text{ is isomorphic to } \zeta\}$, and we denote this set by $cF_k(M, \zeta)$. The extended Chern-Weil construction applies to the set of all k -flat structures on M ; $(\zeta, \nabla) \in F_k(M)$ induces a chain map $\nabla^*: WO_k \rightarrow \Omega^*M$ (or $\nabla^*: W_k \rightarrow \Omega^*M$ if ζ is trivial) whose induced map in cohomology depends only on the concordance class of (ζ, ∇) .

Our discussion has an analog in the category of C^∞ complex vector bundles. The concepts of connection and curvature tensor carry over directly, we just replace the real numbers with the complex numbers throughout. If we let I, I_k, WO_k, W_k also denote their own complexifications, we define the complex Chern-Weil map of a complex connection ∇ on a complex k -plane bundle γ . ∇_x maps I into the complexified de Rham algebra $\Omega_{\mathbb{C}}^*M$, and the classes in the image of the induced map in cohomology are the Chern classes of γ . Complex k -flat structures are defined similarly to real k -flat structures, and $F_k^c(M)$ and $cF_k^c(M)$ will denote the sets of complex k -flat structures, and concordance classes of complex k -flat structures, respectively. We note that, as above, $(\gamma, \nabla) \in F_k^c(M)$ induces a chain map $\nabla_*: WO_k \rightarrow \Omega_{\mathbb{C}}^*M$ (or $\nabla_*: W_k \rightarrow \Omega_{\mathbb{C}}^*M$) whose induced map in cohomology depends only on the concordance class of (γ, ∇) .

The main results of this paper can now be stated more precisely. Let BGl^k denote the classifying space for k -plane bundles [9]. It is well known that a k -plane bundle $\xi \xrightarrow{\pi} M$ determines a classifying map $\xi: M \rightarrow BGl^k$, and that there is a canonical bijection between homotopy classes of maps $[M, BGl^k]$, and isomorphism classes of k -plane bundles over M .

THEOREM 3.5. *There is a fibration of CW complexes $B\pi^k: BC^k \rightarrow BGl^k$ with the following properties.*

(i) *A k -flat structure (ξ, ∇) determines a lifting of the classifying map of ξ to BC^k*

$$\begin{array}{ccc} & & BC^k \\ & \nearrow^{(\xi, \nabla)} & \downarrow B\pi^k \\ M & \xrightarrow{\xi} & BGl^k \end{array}$$

(ii) *There is a canonical bijection between concordance classes of k -flat structures on M and homotopy classes of maps $[M, BC^k]$.*

We are also able to show a relationship between the topology of the fibration $B\pi^k: BC^k \rightarrow BGl^k$ and the cohomology of W^k . Let FC^k denote the homotopy fibre of BC^k .

PROPOSITION 3.6. $\pi_i FC^k = 0, 1 \leq i \leq 2k$.

THEOREM 4.10. *There is a subcomplex $V_k \subset W_k$ (see §4) such that $i_k: H^*V_k \rightarrow H^*W_k$ is injective.*

*Furthermore, there is an injective map $H^*V_k \rightarrow H^*FC_k$.*

2. We will carry out our constructions in the category of semisimplicial sets. This section contains standard technical results which we will need. For proofs and further details, please refer to [8, 10].

A pseudogroup Γ is a category in which each morphism is invertible and the class of all morphisms forms a set. If Γ_1 and Γ_2 are pseudogroups, a map from Γ_1 to Γ_2 is a functor from Γ_1 to Γ_2 . Source and target maps σ and τ are defined in the obvious way, and composition of functions induces a multiplication on the set $\{(g, g') \in \text{mor } \Gamma \times \text{mor } \Gamma \mid \sigma(g') = \tau(g)\}$. If $\text{obj } \Gamma$ consists of a single element, Γ is a group.

Haefliger [5] extends the theory of group structures and classifying spaces in the topological category to include arbitrary pseudogroups. We will make a similar generalization in a semisimplicial setting.

Let $\Delta_n = \{(t_0, t_1, \dots, t_n) \in R^{n+1} \mid \sum t_i = 1\}$ be the n -simplex, and let SS be the category whose objects are $\{\Delta_n \mid n \geq 0\}$ and whose morphisms are affine extensions of order preserving vertex maps. If C is a category (i.e., sets, groups, pseudogroups) a semisimplicial (ss) object in C is a contravariant functor from SS to C , and a map of ss objects is a natural transformation of functors.

Let Γ be an ss pseudogroup; then each Γ_n is a pseudogroup and $\partial_i: \Gamma_n \rightarrow \Gamma_{n-1}$ and $s_i: \Gamma_n \rightarrow \Gamma_{n+1}$ are functors satisfying the usual relations. Note that $\text{obj } \Gamma$ and $\text{mor } \Gamma$ are ss sets and $\sigma, \tau: \text{mor } \Gamma \rightarrow \text{obj } \Gamma$ are ss maps.

Let \mathcal{S} be the category of ss sets and suppose $S \in \mathcal{S}$. A $\Gamma - 1$ -cochain ψ on S is a collection of ss maps $\psi(\alpha): \tilde{\alpha} \rightarrow \text{mor } \Gamma$ for $\alpha \in S$ such that, for each pair α, β and each $t \in \tilde{\alpha} \cap \tilde{\beta}$, there is a $g \in \Gamma$ for which $\sigma(\psi(\alpha)(t)) = \sigma(g \circ \psi(\beta)(t))$. $\psi(s_i \alpha)(s_i \alpha) = s_i(\psi(\alpha)(\alpha))$ for all degeneracies s_i .

A $\Gamma - 2$ -cochain ϕ on S is a collection of ss maps $\phi(\alpha, \beta): \tilde{\alpha} \cap \tilde{\beta} \rightarrow \text{mor } \Gamma$ for $\alpha, \beta \in S$ such that if $t \in \tilde{\alpha} \cap \tilde{\beta} \cap \tilde{\gamma}$, $\sigma\phi(\alpha, \beta)(t) = \tau\phi(\beta, \gamma)(t)$. ϕ is a *cocycle* if $\phi(\alpha, \gamma)(t) = \phi(\alpha, \beta)(t) \circ \phi(\beta, \gamma)(t)$ whenever $t \in \tilde{\alpha} \cap \tilde{\beta} \cap \tilde{\gamma}$. The cocycle condition implies $\phi(\alpha, \alpha)(t) = \text{id}(\sigma\phi(\alpha, \alpha)(t)) = \text{id}(\tau\phi(\alpha, \alpha)(t))$ and $\phi(\alpha, \beta)(t) = (\phi(\beta, \alpha)(t))^{-1}$. A cocycle ϕ is normalized if $s_i(\phi(\alpha, \alpha)(\alpha)) = \phi(s_i \alpha, s_i \alpha)(s_i \alpha)$ for each degeneracy s_i .

If ψ is a $\Gamma - 1$ -cochain and ϕ is a Γ -cocycle such that $\sigma\psi(\alpha)(\alpha) = \tau\phi(\alpha, \alpha)(\alpha) = \sigma\phi(\alpha, \alpha)(\alpha)$, then $\psi \cdot \phi$ is the Γ -cocycle given by

$$\psi \cdot \phi(\alpha, \beta)(t) = \psi(\alpha)(t) \circ \phi(\alpha, \beta)(t) \circ \psi(\beta)(t)^{-1}.$$

Note that $\psi \cdot \phi$ is normalized if both ψ and ϕ are normalized, and $\phi_1 \sim \phi_2 \leftrightarrow \phi_1 = \psi \cdot \phi_2$ is an equivalence relation.

Let $\hat{\Gamma}(S)$ be the set of normalized $\Gamma - 1$ -cochains on S , $\bar{\Gamma}(S)$ = set of normalized Γ -cocycles on S and $\Gamma(S) = \bar{\Gamma}(S)/\sim$. An element of $\Gamma(S)$ is a Γ -structure on S .

If $f: S \rightarrow T$ is an ss map, there are induced maps $\hat{\Gamma}(f): \hat{\Gamma}(T) \rightarrow \hat{\Gamma}(S)$ and $\bar{\Gamma}(f): \bar{\Gamma}(T) \rightarrow \bar{\Gamma}(S)$ defined by

$$\begin{aligned} \hat{\Gamma}(f)(\psi)(\alpha)(t) &= \psi(f(\alpha))(f(t)), \\ \bar{\Gamma}(f)(\phi)(\alpha, \beta)(t) &= \phi(f(\alpha), f(\beta))(f(t)). \end{aligned}$$

Since $\bar{\Gamma}(f)(\psi \cdot \phi) = \hat{\Gamma}(f)(\psi) \cdot \bar{\Gamma}(f)(\phi)$, $\bar{\Gamma}(f)$ induces a map $\Gamma(f): \Gamma(T) \rightarrow \Gamma(S)$.

Define an ss set $B\Gamma$ where $B\Gamma_n = \bar{\Gamma}(\Delta_n)$ and $\partial_i: B\Gamma_n \rightarrow B\Gamma_{n-1}$ and $s_i: B\Gamma_n \rightarrow B\Gamma_{n+1}$ are defined by $\partial_i \alpha = \bar{\Gamma}(\gamma_i)(\alpha)$ and $s_i \alpha = \bar{\Gamma}(p_i)(\alpha)$.

PROPOSITION 2.2. *There is a Γ -cocycle $\bar{\omega} \in \bar{\Gamma}(B\Gamma)$ such that the natural transformation $\bar{\mathcal{T}}: \mathcal{S}(S, B\Gamma) \rightarrow \bar{\Gamma}(S)$ defined by $\bar{\mathcal{T}}(f) = \bar{\Gamma}(f)(\bar{\omega})$ is a bijection.*

In other words, to each cocycle $\phi \in \bar{\Gamma}(S)$, there is a unique classifying map $\hat{\phi}: S \rightarrow B\Gamma$, such that $\bar{\Gamma}(\hat{\phi})(\bar{\omega}) = \phi$. $B\Gamma$ acts as a classifying complex for Γ -structures on simplicial sets (i.e., ss sets generated by simplicial complexes). Let $\bar{\mathcal{S}} \in \mathcal{S}$ be the subcategory of simplicial sets. If $S \in \bar{\mathcal{S}}$, a *subdivision* of S is the simplicial set generated by a subdivision of the underlying simplicial complex. Let $|\cdot|: \mathcal{S} \rightarrow \text{CW complexes}$ be the geometric realization functor [8]. Then it follows from the simplicial approximation theorem as proved in [10] that for any $S \in \bar{\mathcal{S}}$ and any homotopy class of maps $[h]$ from $|S|$ to $|B\Gamma|$, there is a subdivision S' of S and a semisimplicial map $g: S' \rightarrow B\Gamma$ such that $|g| \simeq h: |S| = |S'| \rightarrow |B\Gamma|$.

Suppose S' is a subdivision of S . Γ -structures $\Phi \in \Gamma(S)$ and $\Phi' \in \Gamma(S')$ are *concordant* if there are cocycles $\phi \in \Phi$ and $\phi' \in \Phi'$ such that $|\hat{\phi}| \simeq |\hat{\phi}'|: |S| (= |S'|) \rightarrow |B\Gamma|$. Let $c\Gamma(S) = \bigcup_{S'} \{\Gamma(S') | S' \text{ is a subdivision of } S\} / \sim$, where \sim denotes concordance.

The universal Γ -structure $\omega \in \Gamma(B\Gamma)$ is the Γ -structure represented by the Γ -cocycle $\bar{\omega} \in \bar{\Gamma}(B\Gamma)$ constructed in 2.2. Note that if $|f|: |S| \rightarrow |B\Gamma|$ is represented by an ss map $f: S' \rightarrow B\Gamma$, the concordance class of $\Gamma(f)(\omega)$ depends only on the homology class of $|f|$.

THEOREM 2.3. *The natural transformation $\mathcal{T}: [|S|, |B\Gamma|] \rightarrow c\Gamma(S)$ defined by $\mathcal{T}([|f|]) = [\Gamma(f)(\omega)]$ is a bijection for all $S \in \bar{\mathcal{S}}$.*

REMARK. If $B\Gamma$ is a Kan complex [8], we can write a simpler formulation of 2.3 entirely within the category of ss sets. In this case, we call Γ -structures $\Phi_0, \Phi_1 \in \Gamma(S)$ concordant if their classifying maps $\hat{\phi}_0, \hat{\phi}_1: S \rightarrow B\Gamma$ are homotopic. According to [8], this is an equivalence relation on $\Gamma(S)$ and there is a bijection $\Gamma(S)/\sim \leftrightarrow [S, B\Gamma]$.

We describe below a sufficient condition on Γ for which $B\Gamma$ is a Kan complex.

An ss pseudogroup Γ is *Kan* if $\text{obj } \Gamma$ and $\text{mor } \Gamma$ are Kan complexes and both $\sigma, \tau: \text{mor } \Gamma \rightarrow \text{obj } \Gamma$ are Kan fibrations.

THEOREM 2.4. *If Γ is Kan, $B\Gamma$ is a Kan complex.*

COROLLARY 2.5. *If Γ is Kan and S is an ss set, homotopy is an equivalence relation on $\mathcal{S}(S, B\Gamma)$.*

The final result we will need is a relative version of Theorem 2.3. Suppose Γ^1 and Γ^2 are ss pseudogroups. A morphism $\pi: \Gamma^1 \rightarrow \Gamma^2$ is a collection of functors $\{\pi_n: \Gamma_n^1 \rightarrow \Gamma_n^2\}$ commuting with the boundary and degeneracy maps. We call $\pi: \Gamma^1 \rightarrow \Gamma^2$ a *Kan map* if $\pi: \text{obj } \Gamma^1 \rightarrow \text{obj } \Gamma^2$ is a Kan fibration and if, for each $\gamma \in \text{mor } \Gamma^2$ and each $x \in \text{obj } \Gamma^1$ such that $\sigma^2 \gamma = \pi(x)$, there is some $\gamma' \in \text{mor } \Gamma^1$ such that $\sigma^2 \gamma' = x$ there is some $\gamma' \in \text{mor } \Gamma^1$ such that $\pi(\gamma') = \gamma$.

If ϕ is a Γ^1 -cocycle on S , $\bar{\pi}_* \phi$ is the Γ^2 -cocycle given by $(\bar{\pi}_* \phi)(\alpha, \beta)(s) = \pi(\phi(\alpha, \beta)(s))$ and if ψ is a $\Gamma^1 - 1$ -cochain on S , $\hat{\pi}_* \psi \in \Gamma^2(S)$ is defined similarly. Note $\bar{\pi}_*(\psi \cdot \phi) = \hat{\pi}_*(\psi) \cdot \bar{\pi}_*(\phi)$; hence, there is an induced map $\pi_*: \Gamma^1(S) \rightarrow \Gamma^2(S)$. Let $E\pi: E\Gamma^1 \rightarrow E\Gamma^2$ and $B\pi: B\Gamma^1 \rightarrow B\Gamma^2$ be the ss maps such that $E\pi_n = \hat{\pi}_n^*$ and $B\pi_n = \bar{\pi}_n^*$.

Note that π_* preserves the concordance relation.

THEOREM 2.6. 1. *The natural transformations $\mathcal{I}': [|S|, |B\Gamma^1|] \leftrightarrow c\Gamma^1(S)$ restrict to a bijection $[|\hat{\theta}|, |B\pi|] \leftrightarrow c\Gamma^1(S, \theta)$.*

2. *If $\pi: \Gamma^1 \rightarrow \Gamma^2$ is a Kan map, then $B\pi: B\Gamma^1 \rightarrow B\Gamma^2$ is a Kan fibration.*

3. It is our aim in this section to construct a classifying space for concordance classes of k -flat structures on manifolds. We show that there is a family of CW complexes $|BC^k|$ for $k \geq 1$ and fibrations $|B\pi^k|: |BC^k| \rightarrow |BG^k|$ which satisfy the following property. Given a C^∞ vector bundle ξ over M , there is a one-to-one correspondence between the set of homotopy classes of liftings $[\hat{\xi}, |B\pi^k|]$ and the set of concordance classes $cF_k(M, \xi)$

$$\begin{array}{ccc} & & |BC^k| \\ & \nearrow & \downarrow |B\pi^k| \\ M & \xrightarrow{\quad} & |BG^k|. \end{array}$$

The outline is as follows: we construct a map of ss pseudogroups $\pi^k: C^k \rightarrow G^k$, where G^k is the ss group adjoint to $\text{Sing}^\infty G^k$ (i.e., G^k is the set of C^∞ vector bundle isomorphisms $\{g: \Delta_n \times \mathbf{R}^k \rightarrow \Delta_n \times \mathbf{R}^k\}$ covering the identity map). A G^k structure on a simplicial set S is just a vector bundle on S . Theorem 2.6 implies that there is an ss map $B\pi^k: BC^k \rightarrow BG^k$ such that, given a G^k structure $\bar{\gamma}$ on S , there is a bijection between $[|\hat{\gamma}|, |B\pi^k|]$ and $cC^k(S, \bar{\gamma})$.

On the other hand, we show that to each manifold, we can associate a simplicial complex L and a homotopy equivalence $i: M \rightarrow |L|$ which induces a bijection between $cF_k(M, |\bar{\gamma}|)$ and $cC^k(\tilde{L}, \bar{\gamma})$. If we choose $\bar{\gamma}$ so that $|\hat{\gamma}| = \hat{\xi}$, the classification theorem is a consequence of the following chain of bijections: $cF_k(M, \xi) \leftrightarrow cC^k(L, \bar{\gamma}) \leftrightarrow [|\hat{\gamma}|, |B\pi^k|] \leftrightarrow [\hat{\xi}, |B\pi^k|]$.

Let $\omega \times \mathbf{R}^k$ denote the trivial bundle $\omega \times \mathbf{R}^k \rightarrow \omega$ for any space ω , and if $\omega' \subset \omega$, we identify $\omega' \times \mathbf{R}^k$ as a subspace of $\omega \times \mathbf{R}^k$. Furthermore, if $f: V \rightarrow \omega$ is continuous, $f^*(\omega \times \mathbf{R}^k)$ is identified with $V \times \mathbf{R}^k$ via $(x, f(x), v) \sim (x, v)$. Finally we assume all bundle maps from $\omega \times \mathbf{R}^k$ to itself cover the identity.

A connection on $\Delta_n \times \mathbf{R}^k$ is an \mathbf{R} -linear map $\nabla: C^\infty(\tau_*\Delta_n) \times C^\infty(\Delta_n \times \mathbf{R}^k) \rightarrow C^\infty(\Delta_n \times \mathbf{R}^k)$ satisfying the usual conditions, where Δ_n is viewed as a smooth manifold with corners.

Define an ss set θ^k where $\theta_n^k = \{k\text{-flat connections on } \Delta_n \times \mathbf{R}^k\}$, and boundaries and degeneracies are given by $\partial_i \Delta = \gamma_i^* \nabla$ and $s_i \nabla = \rho_i^* \nabla$. G^k will denote the ss group whose n -simplexes are C^∞ bundle maps from $\Delta_n \times \mathbf{R}^k$ to itself and C^k is the ss pseudogroup such that $\text{obj } C^k = \theta^k$ and $\text{mor } C^k = \{(\nabla, \nabla', g) | \nabla, \nabla' \in \theta^k, g \in G^k \text{ and } g^* \nabla' = \nabla\}$. Source and target maps are defined in the obvious way, and there is a map of ss pseudogroups $\pi^k: C^k \rightarrow G^k$ given by $\pi(\nabla, \nabla', g) = g$.

THEOREM 3.1. *The map $|B\pi^k|: |BC^k| \rightarrow |BG^k|$ is a fibration such that, given a simplicial set S and a G^k -structure $\bar{\gamma} \in G^k(S)$, there is a one-to-one correspondence between $[|\hat{\gamma}|, |B\pi^k|]$ and $cC^k(S, \bar{\gamma})$.*

PROOF. This is a special case of Theorem 2.6.

Let $h: |L| \rightarrow M$ be a C^∞ triangulation of a manifold M . For each $\alpha \in L$, let $\tau_*\alpha$ be the bundle over $h(|\alpha|)$ of vectors tangent to $h(|\alpha|)$, and if ξ is a k -plane bundle over M , ξ/α will denote the restriction of ξ to $h(|\alpha|)$. Note that if $\beta = \partial_I \alpha$, there is a restriction map $\partial_I: C^\infty(\xi/\alpha) \rightarrow C^\infty(\xi/\beta)$.

A *singular k -flat structure* (skf-structure) on M is a collection $\nabla = (h, \xi, \nabla(\alpha))$ such that $h: |L| \rightarrow M$ is a smooth triangulation, ξ is a C^∞ bundle over M and, for every $\alpha \in L$, $\nabla(\alpha)$ is a k -flat connection on ξ/α such that

$$\nabla(\alpha)(v, s) = \nabla(\beta)(v, s)|_{h(\beta)}$$

whenever $\beta = \partial_I \alpha$ and $v \in T_* \beta$.

If $h(|L|, |K|) \rightarrow (M, \partial M)$ is a relative C^∞ triangulation and $\nabla = (h, \xi, \nabla(\alpha))$ is an skf-structure on M , $\nabla/\partial M$ is an skf-structure on ∂M , and if N is a component of ∂M , ∇/N is defined similarly.

Skf-structures ∇_0 and ∇_1 are *concordant* if there is an skf-structure ∇ on $M \times I$ such that $\nabla|_{M \times \{j\}} = \nabla_j$ for $j = 0, 1$. If $h: |L| \rightarrow M$ is a triangulation, let $SF_k(M, h)$ be the set of skf-structures on M with respect to h , and if $h': |L'| \rightarrow M$ is a subdivision of h (written $h' \leq h$), let

$$cSF_k(M, h) = \bigcup_{h' \leq h} SF_k(M, h')/\text{concordance}.$$

A smooth triangulation $h: |L| \rightarrow M$ induces a map $h_*: F_k(M) \rightarrow SF_k(M)$ defined by $h_*(\zeta, \nabla) = (h, \zeta, \nabla(\alpha))$, where $\nabla(\alpha)$ is the restriction of ∇ to $h(|\alpha|)$. h_* projects to a map $cF_k(M) \rightarrow cSF_k(M, h)$, which we also denote by h_* .

Fix a locally finite, smooth triangulation $h: |L| \rightarrow M$, and let \tilde{L} be the ss set generated by L .

LEMMA 3.2. *There is a bijection $cSF_k(M, h) \leftrightarrow cC^k(\tilde{L})$.*

PROOF. Let $h^0: |L^0| \rightarrow M$ be a subdivision of h and let $\nabla^0 = (h^0, \zeta^0, \nabla^0(\alpha)) \in SF_k(M, h')$ for any m -simplex α of L^0 , $(\zeta^0|_\alpha, \nabla^0(\alpha))$ is a k -flat structure on $h^0(|\alpha|)$ so we can choose a trivialization $\bar{\alpha}: (\nabla_n \times \mathbf{R}^k, \bar{\nabla}(\alpha)) \rightarrow (\zeta^0|_\alpha, \nabla^0(\alpha))$ covering $h^0 \circ \hat{\alpha}$.

If $\alpha \in L_m^0$, $\beta \in L_p^0$, $s = \partial_I \alpha = \partial_J \beta \in \alpha \cap \beta$, let $\phi^0 \in \bar{C}^k(L^0)$ be the cocycle given by

$$\phi^0(\alpha, \beta)(s) = (\partial_I \bar{\nabla}(\alpha), \partial_J \bar{\nabla}(\beta), \hat{\alpha}^{-1} \circ \beta|_{\partial_J \Delta_p}).$$

If ϕ' comes from a second choice of trivialization, the change in trivialization produces a 1-chain ψ such that $\phi' = \psi + \phi^0$, therefore, yields a concordant cocycle; hence, we have a well-defined map $f: \bigcup_{h' \leq h} SF_k(M, h') \rightarrow \bigcup_{h' \leq h} C^k(\tilde{L}')$. Then f projects to a map $F: cSF_k(M, h) \rightarrow cC^k(\tilde{L})$. If $\nabla^1 \in SF_k(M, h^1)$ is concordant to ∇^0 , there is an skf-structure $\nabla \in SF_k(M \times I, \tilde{h})$ such that $\nabla|_{M \times \{j\}} = \nabla^j$ for $j = 0, 1$. As above, we define cocycles $\phi \in \bar{C}^k(\tilde{L})$, $\phi^0 \in \bar{C}^k(\tilde{L}^0)$, $\phi^1 \in \bar{C}^k(\tilde{L}^1)$, so that $|\phi|: M \times I \rightarrow |BC^k|$ is a homotopy between $|\phi^0|$ and $|\phi^1|$. Thus, $f(V^0)$ is concordant to $\Gamma(V^1)$, and $\Gamma: |V| \rightarrow |\Gamma(V)|$ is well defined. We will show F is a bijection.

Surjectivity. Let $\phi \in \bar{C}^k(L')$ where $h': |L'| \rightarrow M$ subdivides h . Consider the k plane bundle $\zeta = \coprod_{\alpha \in L'} h' \circ \partial(\Delta_n) \times \mathbf{R}^k / \sim$ where we identify

$$(h' \circ \hat{\alpha}(S_I x), V) \sim (h' \circ \hat{\beta}(x), \pi(\phi_{\beta\alpha}(\beta)(x)(V))) \quad \forall \beta = \partial_I \alpha.$$

ζ can be given a unique smooth structure and we define an skf-structure $\nabla = (h', \zeta, \nabla(\alpha))$ on M as follows: each $\nabla(\alpha)$ is determined by the property that if $f(\alpha): \Delta_m \times \mathbf{R}^k \rightarrow \zeta|_\alpha$ is the composite

$$\Delta_m \times \mathbf{R}^k \xrightarrow{h' \circ \hat{\alpha} \times 1} h' \circ \hat{\alpha}(\Delta_m) \times \mathbf{R}^k \xrightarrow{i} \coprod_{\alpha \in L'} h' \circ \hat{\alpha}(\Delta_m) \times \mathbf{R}^k \rightarrow \zeta,$$

then $f(\alpha): (\Delta_m \times \mathbf{R}^k, \phi(\alpha, \alpha)(\alpha)) \rightarrow (\zeta|_\alpha, \nabla(\alpha))$ is a map of k -flat structures. $f([\phi]) = \nabla$ by construction, hence F is surjective.

Injectivity. Suppose $[\nabla_0], [\nabla_1] \in cSF_k(M, h)$ and $F([\nabla_0]) = F([\nabla_1])$. Then we can find cocycle representatives $\phi_0 \in \overline{C}^k(\tilde{L}^0)$, $\phi_1 \in \overline{C}^k(\tilde{L}^1)$ representing $F([\nabla_0])$ and $F([\nabla_1])$, and a homotopy $\psi: M \times I \rightarrow |BC^k|$ between $|\phi_0|$ and $|\hat{\phi}_1|$.

According to [14, Theorem 10.6, p. 101], given a bounded manifold N and a smooth triangulation $f: |K| \rightarrow \partial N$, there is a relative smooth triangulation $\underline{f}: (|\underline{K}|, |K|) \rightarrow (N, \partial N)$ extending f .

Let $h_0: |L^0| \rightarrow M$, $h_1: |L^1| \rightarrow M$ be the subdivisions of h corresponding to ∇_0 and ∇_1 . Choose a smooth triangulation

$$h: (|\underline{L}|, |L^0|, |L^1|) \rightarrow (M \times I, M \times \{0\}, M \times \{1\})$$

extending $h_0 \cup h_1$. By the relative simplicial approximation theorem [10, Theorem 5.1, p. 328], there is a relative subdivision

$$\underline{h}': (|\underline{L}'|, |L^0|, |L^1|) \rightarrow (M \times I, M \times \{0\}, M \times \{1\})$$

and a simplicial map $\theta: \underline{L}' \rightarrow BC^k$ such that $|\theta| \simeq \gamma \text{ rel } \partial M \times I$. Therefore, $\theta|_{L_j} = \hat{\theta}_j$ for $j = 0, 1$.

As above, we construct $\nabla \in SF_k(M, \underline{h}')$ such that $f(\nabla) = \Gamma(\theta)(w) \in C^k(\tilde{L}')$. Then $\nabla|_{M \times \{j\}} = \nabla_j$ for $j = 0, 1$; hence ∇_0 is concordant to ∇_1 and F is injective.

To prove Theorem 3.5, we must show that for a fixed triangulation h , there is a bijection between the set of concordance classes of skf-structures on M (with respect to h) and the set of concordance classes of k -flat structures on M . The bijection is the result of a smoothing procedure which we describe below.

Let $\mathcal{U} \subset M$ be an open set. An skf-structure $\nabla = (h, \xi, \nabla(\alpha)) \in SF_k(M, h)$ is *smooth on \mathcal{U}* if there is a C^∞ k -flat structure $(\hat{\xi}, \hat{\nabla})$ on \mathcal{U} such that

$$(\hat{\xi}, \hat{\nabla})/(\mathcal{U} \cap h|\alpha|) = (\xi/\alpha, \nabla(\alpha)/\mathcal{U} \cap h|\alpha|)$$

for all $\alpha \in L$. Note that an skf-structure ∇ which is smooth on a neighborhood of the $(n-1)$ skeleton of an n -manifold M is smooth on M .

In the next lemma, we use an inductive procedure to smooth an skf-structure over the skeleta of a triangulation. We assume first that M is an open subset of Euclidean n -space; this enables us to choose a triangulation of M which is compatible with the affine structure of \mathbf{R}^N .

We then construct neighborhoods \mathcal{U}_d of the $(d-1)$ -skeleton, and smooth maps $G: \mathcal{U}_d \rightarrow \mathcal{U}_d$ which are the identity near the boundary of \mathcal{U}_d , and which project smaller neighborhoods $\mathcal{U}'_d \subset \mathcal{U}_d$ onto the d -skeleton. We pull back the k -flat structure with these maps, the result is a k -flat structure which is smooth on a neighborhood of the d -skeleton.

LEMMA 3.3. *Suppose $M \subset \mathbf{R}^N$ is an open subset and $h: |L| \rightarrow M$ is a locally finite smooth triangulation such that $h|\alpha|$ is an affine subset of \mathbf{R}^N for all α . If $\nabla = (h, \xi, \nabla(\alpha)) \in SF_k(M, h)$ and ∇ is smooth on an open neighborhood \mathcal{U} of a subcomplex $h(|K|) \subset M$ such that $h(|K|) \subset \mathcal{U}$, then ∇ is concordant to an skf-structure ∇' which is smooth on M and which agrees with ∇ on some smaller neighborhood \mathcal{U}' of $h(|K|)$.*

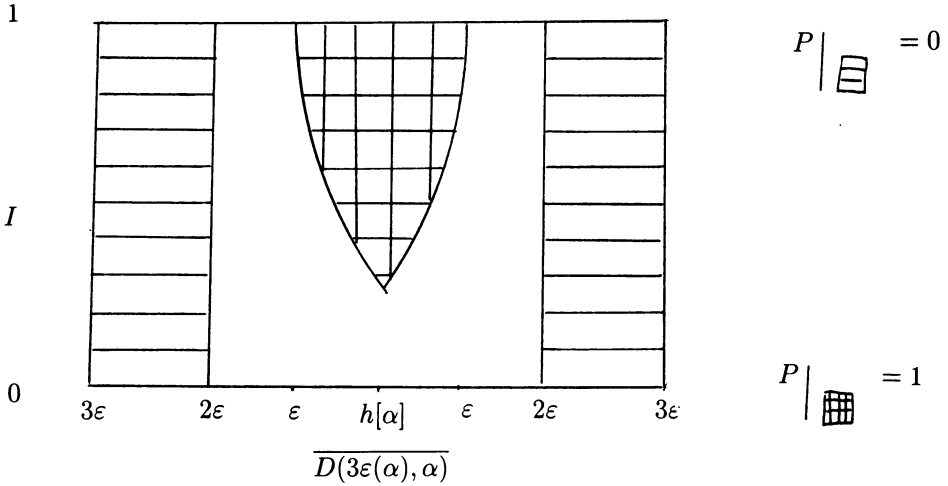
PROOF. We will use an inductive argument to prove an absolute version of 3.3 and then indicate the changes needed for the relative statement. Let $\alpha \in L$.

Recall:

$$\begin{aligned}\text{St } \alpha &= \{\gamma \in L \mid \exists \beta \in L, \alpha \subset \beta, \gamma \subset \beta\}, \\ \text{Lk } \alpha &= \{\beta \in \text{St } \alpha \mid \alpha \not\subset \beta\}.\end{aligned}$$

Step 0. If $\varepsilon > 0$ and $\alpha \in L$ is a 0-simplex, let $D(\varepsilon, \alpha) = \{x \in \mathbf{R}^N \mid |x - h(\alpha)| < \varepsilon\}$. For every 0-simplex α , let $\varepsilon(\alpha) > 0$ be a number such that $D(3\varepsilon(\alpha), \alpha) \subset h|\text{St } \alpha| - h|\text{Lk } \alpha|$ and if β is any other 0-simplex, $\overline{D(3\varepsilon(\alpha), \alpha)} \cap \overline{D(3\varepsilon(\beta), \beta)} = \emptyset$. Let $\rho(\alpha): D(\varepsilon(\alpha), \alpha) \rightarrow h|\alpha|$ be the projection map, and $g: I \rightarrow I$ a smooth map that takes a neighborhood of ∂I into itself. Suppose $\rho: \bigcup_{\alpha} D(3\varepsilon(\alpha), \alpha) \times I \rightarrow I$ is a C^{∞} map such that

- (i) $\rho(m, t) = 0$ if $t = 0$ or if $|m - h(\alpha)| \geq 2\varepsilon(\alpha)$,
- (ii) $\rho(m, t) = 1$ if $g(t) > 0$ and $|m - h(\alpha)| \leq g(t) \cdot t(\alpha)$.



Let $G: \bigcup_{\alpha} D(3\varepsilon(\alpha), \alpha) \times I \rightarrow \bigcup_{\alpha} D(3\varepsilon(\alpha), \alpha)$ be given by

$$G(m, t) = (1 - \rho(m, t)) \cdot m + \rho(m, t)h(\alpha).$$

G is a smooth map and $G/\bigcup_{\alpha} D(3\varepsilon(\alpha), \alpha) - D(2\varepsilon(\alpha), \alpha) \times I$ is projection on the first factor. Define $H: M \times I \rightarrow M$ by

$$\begin{aligned}H(m, t) &= G(m, t), & m \in \bigcup_{\alpha} D(3\varepsilon(\alpha), \alpha) \\ &= m, & m \in M \setminus \bigcup_{\alpha} D(2\varepsilon(\alpha), \alpha).\end{aligned}$$

H is a smooth map since it is given as the union of two smooth maps over an open set. Note that $H/(M \times \{0\}) = \text{id}$, $H/(D(\varepsilon(\alpha), \alpha) \times \{1\}) = p(\alpha)$, and for all $\gamma \in L$, $H(h|\gamma| \times I) \subset h|\gamma|$. For suppose $m \in h|\gamma|$; if $m \notin \bigcup_{\alpha} D(2\varepsilon(\alpha), \alpha)$, $H(m, t) = m \in h|\gamma|$ and if $m \in D(2\varepsilon(\alpha), \alpha)$ for some 0-simplex α , then $\alpha < \gamma$ and

$$H(m, t) = (1 - \rho(m, t)) \cdot m + \rho(m, t) \cdot h|\alpha| \in h|\gamma|$$

by convexity.

Define an skf-structure $H^*\nabla = (h \times 1, H^*\xi, H^*(\nabla)(\nu))$ on $M \times I$ as follows: $h \times 1: (|L \times \Delta_1|, |L \times \{0\}|, |L \times \{1\}|) \rightarrow (M \times I, M \times \{0\}, M \times \{1\})$ is the product triangulation, and if $\nu \in \gamma \times \Delta_1$ for some $\gamma \in L$, then $H((h \times 1)|\gamma|) \subset h|\gamma|$ and $H^*(\nabla)(\nu) = (H|_{\nu})^*(\nabla(\gamma))$ where Y is any vector field on M tangent to $h|\gamma|$ such that $dH(X(m, t)) = Y(H(m, t))$.

Therefore $\nabla = H^*\nabla/M \times \{0\}$ is concordant to $\nabla' = H^*\nabla/M \times \{1\}$ and ∇' is smooth on $\bigcup_{\alpha} D(\varepsilon(\alpha), \alpha)$ since, for each $\gamma \in L$, and each 0-simplex α ,

$$(\xi'(\gamma), \nabla'(\gamma))|_{h|\gamma| \cap D(\varepsilon(\alpha), \alpha)} = p(\alpha)^*(\xi/\alpha, \nabla(\alpha))|_{h|\gamma| \cap D(\varepsilon(\alpha), \alpha)}.$$

Inductively assume we have constructed an skf-structure $\nabla' = (h, \xi', \nabla'(\alpha))$ concordant to ∇ which is smooth on a neighborhood \mathcal{U} of the $(d-1)$ -skeleton of $h|L|$, and we will suppose that the $(d-1)$ -skeleton is a deformation retract of \mathcal{U} .

For each d -simplex α , let $\alpha', \alpha'', \alpha''' \subset h|\alpha|$ be open subsets, each of which is diffeomorphic to the open d -ball and so that $\alpha' \subset \bar{\alpha}' \subset \alpha'' \subset \bar{\alpha}'' \subset \alpha''' \subset \bar{\alpha}''' \subset$ interior $h|\alpha|$ and $\partial\bar{\alpha}', \partial\bar{\alpha}'', \partial\bar{\alpha}''' \subset \mathcal{U}$.

If $\varepsilon > 0$, let $D(\varepsilon) = \{x \in \mathbf{R}^{N-d} | |x| < \varepsilon\}$. Let $V(\alpha)$ be a neighborhood of $\bar{\alpha}'''$ such that there is a diffeomorphism $\phi(\alpha): \alpha''' \times D(3) \rightarrow V(\alpha)$ which is the identity in the first factor, and affine in the second factor, and so that

$$(i) \quad \overline{V(\alpha)} \cap \overline{V(\beta)} = \emptyset \text{ if } \alpha \neq \beta,$$

$$(ii) \quad \overline{V(\alpha)} \subset h|\text{St } \alpha| - h|\text{Lk } \alpha|.$$

Let $p(\alpha): V(\alpha) \rightarrow \bar{\alpha}'''$ be the projection map, and suppose $g: I \rightarrow I$ is a smooth map that takes a neighborhood of ∂I into itself.

Define C^∞ maps $\rho_1: \bigcup_{\alpha} \bar{\alpha}''' \rightarrow I$ and $\rho_2: D(3) \times I \rightarrow I$ such that

$$(i) \quad \begin{aligned} \rho_1(x) &= 1 & \text{if } x \in \bigcup_{\alpha} \bar{\alpha}', \\ \rho_1(x) &= 0 & \text{if } x \notin \bigcup_{\alpha} \bar{\alpha}'', \end{aligned}$$

$$(ii) \quad \begin{aligned} \rho_2(y, t) &= 0 & \text{if } |y| \geq 2 \text{ or if } t = 0, \\ \rho_2(y, t) &= 1 & \text{if } g(t) > 0 \text{ and } |y| \leq t. \end{aligned}$$

Let $G: \bigcup_{\alpha} \bar{\alpha}''' \times D(3) \times I \rightarrow \bigcup_{\alpha} \bar{\alpha}''' \times D(3)$ be defined by

$$G(x, y, t) = (x, (1 - \rho_2(y, \rho_1(x) \cdot t)) \cdot y).$$

Note that G is a smooth map, and $G(x, y, t) = (x, y)$ if $t = 0$ or $|y| \geq 2$ or $x \in \bigcup_{\alpha} \bar{\alpha}''' - \alpha''$. Let $H: M \times I \rightarrow M$ be defined by

$$\begin{aligned} H(m, t) &= m, & m \in M \bigcup_{\alpha} \phi(\alpha)^{-1}(\alpha'' \times D(2)) \\ &= \phi(\alpha)^{-1}(G(\phi(\alpha)(m, t))), & m \in \text{some } V(\alpha). \end{aligned}$$

H is C^∞ since it is written as the union over open sets of C^∞ maps, and $H/M \times \{0\} = \text{id}$. Furthermore, if $\gamma \in L$, $H(h|\gamma| \times I) \subset h|\gamma|$; for suppose $m \in h|\gamma|$; then if $m \notin \bigcup_{\alpha} V(\alpha)$, we have $H(m, t) = m \in h|\gamma|$, and if $m \in \text{some } V(\alpha)$, then $p(\alpha)(m) \in h|\gamma|$, and if $\phi(\alpha)(m) = (x, y) \in \bar{\alpha}''' \times D(3)$, then

$$\begin{aligned} H(m, t) &= \phi(\alpha)^{-1}(x, (1 - \rho_2(y, \rho_1(x) \cdot t))y) \\ &= \phi(\alpha)^{-1}(x, (1 - \rho_2(y, \rho_1(x, t)) \cdot y + \rho_2(y, \rho_1(x) \cdot \varepsilon)) \cdot 0) \\ &= (1 - \rho_2(y, \rho_1(x, t)))\phi(\alpha)^{-1}(x, y) \\ &\quad + \rho_2(y, \rho_1(x, t))\phi(\alpha)^{-1}(x, 0) \\ &= (1 - \rho_2(y, \rho_1(x, t))) \cdot m + \rho_2(y, \rho_1(x, t)) \cdot p(\alpha)(m) \end{aligned}$$

since $\phi(\alpha)$ is affine. Therefore, $H(m, t) \in h|\gamma|$ by convexity.

Let $H^*\nabla'$ be the induced skf-structure defined as in Step 0 above; then $\nabla' = H^*\nabla'/M \times \{0\}$ is concordant to $H^*\nabla'/M \times \{1\} = \nabla''$. We show ∇'' is smooth on a neighborhood of the d -skeleton of M .

Let $H_1 = H(\cdot, 1): M \rightarrow M$; then $\nabla'' = H_1^*\nabla'$ and by hypothesis, there is a neighborhood \mathcal{U} and a deformation retraction $r: \mathcal{U} \rightarrow (d-1)$ -skeleton, such that

∇' is smooth on \mathcal{U} . Since $\bigcup_\alpha \partial\bar{\alpha}' \subset \mathcal{U}$, it follows that $\bigcup_\alpha h|\alpha| - \alpha' \subset \mathcal{U}$, and if $m \in \bigcup_\alpha h|\alpha| - \alpha'$ then $H_1(m) = m$ so by continuity, there is a neighborhood $\mathcal{U}' \subset \mathcal{U}$ such that $\bigcup_\alpha h|\alpha| - \alpha' \subset \mathcal{U}'$ and $H_1(\mathcal{U}') \subset \mathcal{U}$. H_1 is a smooth map; therefore $H_1^*\nabla' = \nabla''$ must be smooth on \mathcal{U}' .

Finally, let $\mathcal{Z}(\alpha) = \phi(\alpha)^{-1}(\alpha' \times D(1))$ for each α . Note that $H_1|_{\mathcal{Z}(\alpha)} = p(\alpha)/\mathcal{Z}(\alpha): \mathcal{Z}(\alpha) \rightarrow \alpha'$ and for all $\gamma \in L$,

$$(\xi''(\gamma), \nabla''(\gamma))/h|\gamma| \cap \mathcal{Z}(\alpha) = p(\alpha)^*(\xi'(\alpha), \nabla'(\alpha))/h|\gamma| \cap \mathcal{Z}(\alpha)$$

and $\bigcup_\alpha p(\alpha)^*(\xi'(\alpha), \nabla'(\alpha))/\mathcal{Z}(\alpha)$ is a C^∞ k -flat structure on $\bigcup_\alpha \mathcal{Z}(\alpha)$. Therefore, ∇'' is smooth on $\mathcal{U}' \cup \bigcup_\alpha \mathcal{Z}(\alpha)$.

If $m \in d$ -skeleton, then either $m \in \alpha' \subset \mathcal{Z}(\alpha)$ for some α or

$$m \in \left(d\text{-skeleton} \setminus \bigcup_\alpha \alpha' \right) \subset \mathcal{U}.$$

Therefore, $\mathcal{U} \cup \bigcup_\alpha \mathcal{Z}(\alpha)$ is a neighborhood of the d -skeleton.

This completes the inductive step and we conclude that ∇ is concordant to a skf-structure ∇' which is smooth on a neighborhood of the $(N-1)$ -skeleton of M and hence, is smooth on M .

To prove the relative statement, assume ∇ is already smooth on a neighborhood \mathcal{U} of a subcomplex $h|K|$ of M such that $h|K| \subset \overline{h|K|} \subset \mathcal{U}$. Subdivide the triangulation so that there is a subcomplex K' and an open set \mathcal{U}' such that $h|K| \subset \mathcal{U}' \subset h|K'| \subset \mathcal{U}$ and perform the same induction considering only simplexes outside of \mathcal{U}' . Note that the construction implies that the concordance restricted to $\mathcal{U}' \times I$ is the product concordance.

COROLLARY 3.4. *If $M \subset \mathbf{R}^N$ is an open subset, $h: |L| \rightarrow M$ is a triangulation satisfying the hypothesis of 3.3, and ξ is a C^∞ k -plane bundle over M , there is a bijection $cC^k(\tilde{L}, \underline{\gamma}) \leftrightarrow cF_k(M, \xi)$ for $\underline{\gamma} \in G^k(\tilde{L})$ such that $|\underline{\gamma}| \sim h^*\xi$. This is a direct consequence of 3.2 and 3.3.*

We define a transformation of functors $\Phi: [o, |BC^k|] \rightarrow cF_k(\cdot)$ as follows: If $M \subset \mathbf{R}^N$ is open, choose a triangulation $h: |L| \rightarrow M$ satisfying the hypotheses of 3.3, then Φ is defined to be the composite of maps

$$[M, |BC^k|] \xrightarrow{h^*} [|L|, |BC^k|] \xrightarrow{\int} cC^k(\tilde{L}) \xrightarrow{F^{-1}} cSF_k(M, h) \xrightarrow{h^{*-1}} cF_k(M)$$

constructed in 3.1, 3.2 and 3.4.

For arbitrary M , choose an embedding $i: M \rightarrow \mathbf{R}^N$ for $N > 2\dim(M)$, and a tubular neighborhood $p: U \rightarrow M$ of M in \mathbf{R}^N . If we triangulate U by $h: |L| \rightarrow U$ as above, Φ is the composite

$$[M, |BC^k|] \xrightarrow{P^*} [U, |BC^k|] \xrightarrow{\Phi} cF_k(U) \xrightarrow{i^*} cF_k(M).$$

THEOREM 3.5. (i) Φ is a bijection and a natural transformation.

(ii) Let $|B\pi^k|: |BC^k| \rightarrow |BG^k|$ be as in 3.1. If $\zeta \rightarrow M$ is a smooth \mathbf{R}^k vector bundle, Φ restricts to a bijection $[\zeta, |B\pi^k|] \leftrightarrow cF_k(M, \zeta)$.

PROOF. (i) If $M \subset \mathbf{R}^N$ is open, $\Phi: [M, |BC^k|] \rightarrow cF_k(M)$ is a bijection by 3.1 and 3.4. In general, note that $p^*: [M, |BC^k|] \rightarrow [U, |BC^k|]$ is an isomorphism, and since $p \circ i = \text{id}: M \rightarrow M$ and $i \circ p: U \rightarrow U$ is smoothly homotopic to id , there is a bijection $i^*: cF_k(U) \rightarrow cF_k(M)$. Φ does not depend on the choice of embedding or triangulation. Suppose we have embeddings $i_0: M \rightarrow \mathbf{R}^N$, $i_1: M \rightarrow \mathbf{R}^P$, tubular neighborhoods $p_0: U_0 \rightarrow i_0(M)$ and $p_1: U_1 \rightarrow i_1(M)$, and affine triangulations $h_0: |L_0| \rightarrow U_0$ and $h_1: |L_1| \rightarrow U_1$. Without loss of generality we will assume $N = P$.

Choose a one parameter family of embeddings $i_t: M \rightarrow \mathbf{R}^N$ and a one parameter family of tubular neighborhoods $p_t: U_t \rightarrow i_t(M)$ depending smoothly on t , connecting $p_0: U_0 \rightarrow i_0(M)$ and $p_1: U_1 \rightarrow i_1(M)$, and so that $\bigcup_t U_t \in \mathbf{R}^{N+1}$ is an open submanifold. We choose an affine relative triangulation $h: (\underline{L}, L_0, L_1) \rightarrow (U, U_0, U_1)$.

The following commutative diagram shows that Φ is independent of the choices indicated:

$$\begin{array}{ccccc}
 [U_0, |BC^k|] & \xleftarrow{p_0^*} & [M, |BC^k|] & \xrightarrow{p_1^*} & [U_1, |BC^k|] \\
 \downarrow h_0^* & & & & \downarrow h_1^* \\
 [L_0, |BC^k|] & \xleftarrow{j_0^*} & [|\underline{L}|, |BC^k|] & \xrightarrow{j_1^*} & [L_1, |BC^k|] \\
 \downarrow & & & & \downarrow \\
 cC^k(\tilde{L}_0) & \xleftarrow{cC^k(j_0)} & cC^k(\tilde{\underline{L}}) & \xrightarrow{cC^k(j_1)} & cC^k(\tilde{L}_1) \\
 \uparrow F & & \uparrow F & & \uparrow F \\
 cSF_k(U_0, h_0) & & cSF_k(U, \underline{h}) & & cSF_k(U_1, h_1) \\
 \uparrow h_0^* & & \uparrow \underline{h}^* & & \uparrow h_1^* \\
 cF_k(U_0) & \xleftarrow{j_0^*} & cF_k(U) & \xrightarrow{j_1^*} & cF_k(U_1) \\
 & & \uparrow i_0^* = i_1^* & & \\
 & & cF_k(M) & &
 \end{array}$$

where $j_t: U_t \rightarrow U$ denotes the inclusion.

To show Φ is natural, consider first the case where M and N are open subsets of Euclidean space. Let $\alpha: M \rightarrow N$ be a smooth map representing $[\alpha] \in [M, N]$; choose affine triangulation $h_0: |L_0| \rightarrow M$, $h_1: |L_1| \rightarrow N$ and a simplicial approximation $g: L_0 \rightarrow L_1$ to α .

We will show that the following diagram commutes:

$$\begin{array}{ccc}
 [M, |BC^k|] & \xleftarrow{\alpha^*} & [N, |BC^k|] \\
 \downarrow h_0^* & & \downarrow h_1^* \\
 [L_0, |BC^k|] & & [L_1, |BC^k|] \\
 \downarrow & & \downarrow \\
 cC^k(\tilde{L}_0) & \xleftarrow{cC^k(\alpha)} & cC^k(\tilde{L}_1) \\
 \uparrow F & & \uparrow F \\
 cSF_k(M, h_0) & & cSF_k(M, h_1) \\
 \uparrow h_0^* & & \uparrow h_1^* \\
 cF^k(M) & \xleftarrow{\alpha^*} & cF_k(N)
 \end{array}$$

The top square commutes by Theorem 3.1. To show that the bottom square commutes, let $(\zeta, \nabla) \in F_k(N)$. We show that $h_0^* \alpha^*(\zeta, \nabla)$ represents the concordance class $F^{-1} \circ cC^k(\alpha) \circ F \circ h_1^*[(\zeta, \nabla)]$. The latter class is represented by $|g|^* \nabla = (h_0, |g|^* \xi, |g|^* \nabla(\alpha))$, $\alpha \in L_0$, where $\nabla = h_{1*}(\xi, \nabla) \in SF_*(N, h_1)$.

Let $H: M \times I \rightarrow N$ be the homotopy between γ and $|g|$ such that for any simplex $\alpha \in L_0$, $H|_{|\alpha| \times I}$ is linear. Therefore, $H^* \nabla = (h_0 \times \text{id}, H^* \xi, H^* \nabla(\nu))$ is an skf-structure on $H \times I$ with respect to the product triangulation, giving a concordance between $|g|^* \nabla$ and $h_{0*} \circ \gamma^*(\xi, \nabla)$. Hence Φ is natural in this case, and the general case follows directly.

(ii) The relative statement for an open subset $M \subset \mathbf{R}^N$ is a consequence of 3.1 and 3.4. In general, given a smooth $\xi \rightarrow M$, an embedding $i: M \rightarrow \mathbf{R}^N$ and a tubular neighborhood $p: U \rightarrow M$ of M in \mathbf{R}^N , there are bijections $[\hat{\xi}, |B\pi^k|] \leftrightarrow [p^* \xi, |B\pi^k|]$ and $cF_k(M, \xi) \leftrightarrow cF_k(U, p^* \xi)$. Hence, the theorem holds for all manifolds.

Hereafter, we will omit the geometric realization symbols and let $B\Pi^k \rightarrow BGl_k$ denote the fibration of CW complexes. Let FC^k be the homotopy theoretic fibre of $B\Pi^k$. FC^k acts as a classifying space for k -flat structures whose underlying bundle is trivialized.

PROPOSITION 3.6. $\Pi_i FC^k = 0$ if $1 \leq i \leq 2k$.

PROOF. Let $f: S^n \rightarrow FC^k$ be a continuous map. Up to homotopy, f classifies a k -flat structure $(S^n \times \mathbf{R}^k, \nabla)$. $S^n \times \mathbf{R}^k$ sits inside $D^{n+1} \times \mathbf{R}^k$ and we extend ∇ to an affine connection $\tilde{\nabla}$ on $D^{n+1} \times \mathbf{R}^k$ using a partition of unity.

If $P: Gl_k \rightarrow \mathbf{R}$ is an invariant polynomial (see §1) which is homogeneous of degree $l > k$, $P(K_\nabla)$ is a $2l$ form on D^{n+1} ; therefore $P(k_\nabla)$ is identically zero if $n \leq 2k$ since $2l > 2k + 1 \geq n + 1$.

Hence, $\tilde{\nabla}$ is k -flat, and f extends over D^{n+1} .

COROLLARY 3.7. $B\Pi_*^k: \Pi_i BC^k \rightarrow \Pi_i BGl_k$ is an isomorphism if $0 \leq i \leq 2k$.

LEMMA 3.8. $H B\Pi^k: H^i(BGl_k, \mathbf{R}) \rightarrow H^i(BC^k, \mathbf{R})$ is the zero map if $i > 2k$.

PROOF. Suppose the converse. Then there is a nonzero Pontrjagin class $c \in H^i(BC^k, \mathbf{R})$. Let L be a finite polyhedron and $f: L \rightarrow BC^k$ a continuous map such that $Hf(c) \neq 0 \in H^i(L, \mathbf{R})$.

Choose an open manifold \mathcal{U} , and a homotopy equivalence $j: \mathcal{U} \rightarrow L$. Theorem 3.5 implies that $f \circ j$ is homotopic to a map $g: \mathcal{U} \rightarrow BC^k$ that classifies a smooth k -flat structure (ξ, ∇) on \mathcal{U} . Therefore, $g^*(c) \neq 0 \in \text{Pont}(\xi) \subset H^*(\mathcal{U}, \mathbf{R})$, contradicting the k -flatness of ∇ .

In §4, we will consider complex vector bundles as well as real bundles, and the methods developed in this section apply directly to this case.

Define an ss set θ_C^k such that $(\theta_C^k)_n = \{\text{complex } k\text{-flat connections on } \Delta_n \times \mathbf{C}^k\}$, and boundaries and degeneracies are defined by $\partial_i \nabla = \gamma_i^* \nabla$ and $s_i \nabla = \rho_i^* \nabla$. θG_C^k will be the ss group whose n -simplexes are C^∞ bundle maps from $\Delta_n \times \mathbf{C}^k$ to itself, and C_C^k is the ss pseudogroup such that $\text{obj } C_C^k = \theta_C^k$ and $\text{mor } C_C^k = \{(\nabla, \nabla', g) | \nabla, \nabla' \in \theta_C^k, g \in G_C^k \text{ and } g^* \nabla' = \nabla\}$. Let $(\Pi_C^k): C_C^k \rightarrow G_C^k$ be the map of ss pseudogroups defined: $\Pi_C^k(\nabla, \nabla', g) = g$.

THEOREM 3.9. (i) *There is a fibration of CW complexes $B\Pi_{\mathbb{C}}^k: BC_{\mathbb{C}}^k \rightarrow BG_{\mathbb{C}}^k$ ($= BGl(k, \mathbb{C})$) such that, given a C^∞ k -plane bundle ξ over M , there is a bijection $[\hat{\xi}, B\Pi_{\mathbb{C}}^k] \leftrightarrow cF_k^{\mathbb{C}}(M, \xi)$.*

(ii) *Let $FC_{\mathbb{C}}^k$ denote the homotopy fibre of $B\Pi_{\mathbb{C}}^k$. Then $\Pi_i FC_{\mathbb{C}}^k = 0$ if $1 \leq i \leq 2k$.*

(iii) *$H B\Pi_{\mathbb{C}}^k: H^i(BG_{\mathbb{C}}^k, \mathbb{C}) \rightarrow H^i(BC_{\mathbb{C}}^k, \mathbb{C})$ is the zero map if $i > 2k$.*

4. In this section we relate the characteristic classes of foliations to the classifying space for k -flat structures constructed in §4. Following [6], we reformulate the exotic classes as dual homotopy invariants. Techniques from rational homotopy theory are applied to show these invariants can be realized as dual homotopy classes of BC_k and $BC_k^{\mathbb{C}}$, or equivalently, as cohomology classes of FC_k and $FC_k^{\mathbb{C}}$.

We begin by recalling some basic facts about homotopy theory and differential graded (dg) algebras. [11] and [4] will be our references.

Let \mathbf{F} denote either the real or complex numbers. A (graded) algebra over \mathbf{F} is a nonnegatively graded \mathbf{F} -module X together with a skew-commutative multiplication $\mu: X \otimes X \rightarrow X$ and a unit $\eta: \mathbf{F} \rightarrow X$. If $x \in X^m$ and $y \in X^n$, we write $x \cdot y$ for $\mu(x \otimes y) \in X^{m+n}$. (X, d) is a dg algebra if $d: X \rightarrow X$ is a degree 1 map of graded \mathbf{F} -modules satisfying $d^2 = 0$ and $d(x \cdot y) = dx \cdot y + (-1)^{|x|} x \cdot dy$. When there is no chance of confusion, we let X denote the dg algebra (X, d) . \mathbf{F} can be regarded as a dg algebra which is nonzero only in dimension zero, with trivial differential.

An *augmentation* for X is a dg algebra map $\varepsilon: X \rightarrow \mathbf{F}$, $\varepsilon h = \text{id}$ and \bar{X} will denote the augmentation ideal. Let \mathcal{A} and \mathcal{A}_0 be the categories of dg and dga (differential graded augmented) algebras respectively.

If $X \in \mathcal{A}$, the *cohomology* of X is its homology with respect to d . X is *connected* if $H\eta: H^0(\mathbf{F}) \rightarrow H^0(X)$ is an isomorphism and X is *simply connected* if $H^1(X) = 0$. A connected algebra has a unique augmentation which will be used implicitly.

We briefly describe the homotopy theory of dg algebras. Let $X, Y \in \mathcal{A}$, $\mathcal{A}(X, Y)$ will denote the set of dg algebra maps from X to Y and $X \otimes Y$ is the usual tensor product of dg algebras.

Let $\Delta(1, *)$ be the dg algebra generated by indeterminates τ_0 and τ_1 in grade 0, $d\tau_0$ and $d\tau_1$ in grade 1, satisfying relations $\tau_0 + \tau_1 = 1$, and $d(\tau_i) = d\tau_i$ for $i = 0, 1$.

Maps $f_0, f_1 \in (X, Y)$ are *homotopic* if there is a dg algebra map $F: X \rightarrow Y \otimes \Delta(1, *)$ such that $(1 \otimes \partial_1) \circ F = f_i: X \rightarrow Y$ for $i = 0, 1$. Homotopy is not generally an equivalence relation on $\mathcal{A}(X, Y)$.

For each $n \geq 0$, let $X(n) = X$ be the sub-dg algebra generated by $\bigcup_{i=0}^n X^i$ and dX^n . Inductively define subalgebras $X(n, q)$ for $q \geq 0$ as follows: Let $X(n, 0) = X(n-1)$ and if $q > 0$, let $X(n, q)$ be the subalgebra generated by $X(n, q-1)$ and $\{x \in X^n | dx \in X(n, q-1)\}$.

We say that an algebra $M \in \mathcal{A}$ is *minimal* if M is connected, free, and $M(n) = \bigcup_{q \geq 0} M(n, q)$ for all n . A *minimal model* for a connected algebra $X \in \mathcal{A}$ is a map $f: \mu X \rightarrow X$ such that μX is a minimal algebra and f is a cohomology isomorphism.

Minimal models are useful because they often simplify calculations. Fortunately, many algebras can be approximated by minimal algebras.

LEMMA 4.1. *Every connected algebra $X \in \mathcal{A}$ has a minimal model $f: \mu X \rightarrow X$ and μX is unique up to isomorphism [4].*

The *homotopy type* of a connected algebra $X \in \mathcal{A}$ is defined to be the isomorphism class of any minimal model $f: \mu X \rightarrow X$ and the *dual homotopy groups* $\Pi^n X$, $n \geq 0$, are the homology groups of the cochain complex $\mu X / \overline{\mu X} \cdot \mu \overline{X}$.

If $M \in \mathcal{A}$ is minimal, the homotopy relation on the set of maps from M to Y is an equivalence relation for all $Y \in \mathcal{A}$ [4] and we let $[\mu, Y]$ denote the set of homotopy classes of maps from M to Y .

The following results and calculations will be needed later on:

LEMMA 4.2. (i) *If $g: X \rightarrow Y$ is a cohomology isomorphism and M is minimal, $g: [M, X] \rightarrow [M, Y]$ is a bijection [4, 5.7, 5.8].*

(ii) *If $X, Y \in \mathcal{A}$ are simply connected and $Hg: H^*X \rightarrow H^*Y$ is an isomorphism, then $\Pi g: \Pi^*X \rightarrow \Pi^*Y$ is an isomorphism [4].*

There is a close relationship between topological homotopy type and algebraic homotopy type. For any finite CW complex or manifold L , let

$$\Pi^*L = \text{Hom}((\Pi_*L), \mathbf{R}),$$

and if L is an infinite CW complex, then $\Pi^*L = \varprojlim_{\alpha} \Pi^*L_{\alpha}$ as L_{α} ranges over all finite subcomplexes of L .

THEOREM 4.3. *If M is a manifold with $\pi_1(M)$ nilpotent and $\Omega_{\mathbf{F}}^*(M)$ is its de Rham algebra over \mathbf{F} , there is a natural isomorphism $\Pi^*\Omega_{\mathbf{F}}^*(M) \rightarrow \Pi^*M$ [11].*

For more general results along these lines, see [4, 11].

Calculations of rational homotopy groups with minimal models is interesting and instructive. The reader is referred to [15, Chapter 13] for further details. We include below a statement of results which we will need later on.

THEOREM 4.4.

$$(i) \quad \begin{aligned} \Pi^i BO^{2r} &= \mathbf{F}, \quad i = 0, 4, \dots, 4(r-1), 2r, \\ &= 0, \quad \text{all other } i, \end{aligned}$$

$$\begin{aligned} \Pi^i BO^{2r+1} &= \mathbf{F}, \quad i = 0, 4, \dots, 4r, 2r, \\ &= 0, \quad \text{all other } i, \end{aligned}$$

$$(ii) \quad \begin{aligned} \Pi^i BU^n &= \mathbf{F}, \quad i = 0, 2, \dots, 2n, \\ &= 0, \quad \text{all other } i \quad [6]. \end{aligned}$$

Recall from §1, the dg algebra $W_k = I_k \otimes E_{\mathbf{F}}(h_1, h_2, \dots, h_k)$ where

$$I_k \cong P_{\mathbf{F}}^{2k}(c_1, c_2, \dots, c_k).$$

Let $J_k \subset I_k$ be the sub-dg algebra $P_{\mathbf{F}}^{2k}(c_2, c_4, \dots, c_l)$ where l is the largest even integer less than k , and let $V_k = J_k \otimes E_{\mathbf{F}}(h_2, h_4, \dots, h_l)$ where $dh_i = c_i$ and $dc_i = 0$. Let \tilde{I}_k and \tilde{J}_k denote the underlying graded \mathbf{F} -modules of I_k and J_k respectively.

We can regard H^*W_k and H^*V_k as dg algebras with trivial differential. In both cases all products are zero [6]. Let $\mathcal{L}(H^*W_k)$ and $\mathcal{L}(H^*V_k)$ be the free graded Lie algebras [6] on H^*W_k and H^*V_k respectively.

- THEOREM 4.5.** (i) $\Pi^*W_k \cong \text{Hom}(\mathcal{L}(H^*W_k), \mathbf{F})$,
 (ii) $\Pi^*V_k \cong \text{Hom}(\mathcal{L}(H^*V_k), \mathbf{F})$,
 (iii) $\Pi^*I_k \cong \Pi^*W_k \oplus \tilde{I}_k$,
 (iv) $\Pi^*J_k \cong \Pi^*V_k \oplus \tilde{J}_k$ [6].

Recall from §1 that a k -flat structure on a complex k -flat structure (ξ, ∇) on a manifold M induces a chain map $\nabla_*: WO_k + \Omega_{\mathbf{F}}^*M$ whose induced map in cohomology depends only on the concordance class of (ξ, ∇) . Let ∇_* also denote its own restriction to I_k . ∇_* induces a map $\Pi\nabla_*: I_k \rightarrow \Pi^*\Omega_{\mathbf{F}}^*M$, and the following theorem corresponds to Theorem 3.1 of [6]:

THEOREM 4.6. *The map $\Pi\nabla_*: \Pi^*I_k \rightarrow \Pi^*\Omega_{\mathbf{F}}^*M$ depends only on the concordance class of (ξ, ∇) .*

PROOF. Let $(\xi_j, \nabla_j) \in F_k(M \times I)$ and suppose $(\gamma, \hat{\nabla}) \in F_k(M \times I)$ is a concordance between (ξ_0, ∇_0) and (ξ_1, ∇_1) . The truncated Chern-Weil map $\hat{\nabla}_*: I_k \rightarrow \Omega^*(M \times I)$ satisfies $i_j^* \circ \hat{\nabla}_* = \nabla_{j*}$ for $j = 0, 1$. Consider the diagram

$$\begin{array}{ccccc} I_k & \xrightarrow{\hat{\nabla}_*} & \Omega^*(M \times I) & \xrightarrow{\rho^*} & \Omega^*M \\ f \uparrow & & \searrow \mathcal{O} & & \uparrow g \\ \mu I_k & & & & \mu\Omega^*M \end{array}$$

where $f: \mu I_k$ and $g: \mu\Omega^*M \rightarrow \Omega^*M$ are minimal models. By 4.2(i), we can find $\mathcal{O}: \mu I_k \rightarrow \mu\Omega^*M$ such that the diagram homotopy commutes. Therefore $g \cdot \mathcal{O} = i_j^* \circ \rho^* \circ g \circ \mathcal{O} \simeq i_1^* \circ \hat{\nabla}_* \circ f$ or $\nabla_0^* \circ f \simeq \nabla_1^* \circ f$.

REMARK 4.7. (i) The proof of Theorem 4.6 is virtually identical to [6, Theorem 3.1].

(ii) Suppose $f: M \rightarrow N$ is a smooth map and $(\xi, \nabla) \in F_k(N)$. Then the following diagram commutes precisely:

$$\begin{array}{ccc} \Omega^*M & \xrightarrow{f^*} & \Omega^*N \\ (f^*\nabla)_* \swarrow & & \nearrow \nabla_* \\ & I_k & \end{array}$$

Thus, $\Pi\nabla_* \cdot \Pi f^* \circ \Pi(f^*\nabla)_*: \Pi^*I_k \rightarrow \Pi^*\Omega^*N$.

THEOREM 4.8. (i) *There are injective maps from $\pi^*J_k \rightarrow \pi^*BC^k$ and $\pi^*I_k \rightarrow \pi^*BC^k$.*

(ii) *If (ξ, ∇) is a k -flat structure and $f: M \rightarrow BC^k$ is its classifying map as in 3.5, then the diagram*

$$\begin{array}{ccc} \pi^*J_k & \xrightarrow{\nabla_*} & \pi^*M \\ & \searrow & \nearrow f^* \\ & \pi^*BC^k & \end{array}$$

commutes. The analogous statement holds for complex k -flat structures.

PROOF. Let $\rho: BSC^k \rightarrow BC^k$ be the universal covering space of BC^k . We identify BSC^k with the pullback of BC_k over the covering map $\tilde{\rho}: BSO^k \rightarrow BO^k$. For any k , let $BSO_n^{(k)}$ denote the n -skeleton of BSO^k .

First suppose $k = 2r + 1$. By Proposition 3.6 the inclusion $\tilde{j}: BSO_k^{(2k)} \rightarrow BSO_k$ lifts as shown:

$$\begin{array}{ccccc}
 & & FC^k & & FC^k \\
 & \nearrow j & \downarrow & & \downarrow \\
 & & BSC^k & \xrightarrow{\bar{\rho}} & BC^k \\
 & & \downarrow & & \downarrow \\
 BSO_{(2k)}^k & \xrightarrow{\tilde{y}} & BSO^k & \xrightarrow{\tilde{\rho}} & BO^k
 \end{array}$$

Choose a homotopy equivalence $h: \mathcal{U} \rightarrow BSO_{(2k)}^k$ where \mathcal{U} is an open manifold. By Theorem 3.5, $\rho \circ j \circ h$ classifies (up to homotopy) a k -flat structure (ξ, ∇) on \mathcal{U} . Let $\nabla_*: I_k \rightarrow \Omega_{\mathbf{R}}^* \mathcal{U}$ be the induced map and let ∇_* also denote its own restriction to J_k . By construction,

$$\nabla_*: H^* J_k \rightarrow P_{\mathbf{R}}^{2k}(C_1, \dots, C_r) \rightarrow H^* \mathcal{U} \xrightarrow{(h^*)^{-1}} H^* BSO_{(2k)}^k$$

is an isomorphism. It now follows from Lemma 4.2 that $\Pi \nabla_*: \Pi^* J_k \rightarrow \Pi^* \mathcal{U} \cong \Pi^* BSO_{(2k)}^k$ is an isomorphism as well. The universal properties of BSC^k together with Remark 4.7 imply that $\Pi^* J_k$ injects into $\Pi^* BSC^k$ as well.

Since $\rho^*: \Pi^* BC^k \rightarrow \Pi^* BSC^k$ is an isomorphism in dimensions ≥ 2 , it follows that $\Pi^* J_k$ also injects into $\Pi^* BC^k$.

If $k = 2r$, we replace $BSO_{(2k)}^k$ with $BSO_{(2k)}^{k-1}$ and argue as above. In this case, $\Phi_*: H^* J_k \cong P^{2k}(C_1, \dots, C_n) \rightarrow H^* BSO_{(2k)}^{k-1}$ is an isomorphism.

To prove (ii), suppose we are given $(\xi, \nabla) \in F_k(M)$ and its classifying map $f: M \rightarrow BC^k$. We will assume ξ is oriented. Then f lifts to a map $\tilde{f}: M \rightarrow BSC^k$. Since M is finite dimensional, we can assume, up to homotopy, that $\tilde{f}(M)$ is contained in a finite dimensional subcomplex $K \subset BSC^k$. We will also suppose that the $2k$ skeleton of BSC^k is contained in K . Choose open manifolds U, V , homotopy equivalences $l: K \rightarrow V$, $j: BSC_{(2k)}^k \rightarrow U$, and a smooth map $g: U \rightarrow V$ homotopic to $l \circ i \circ j^{-1}$ where $i: BSC_{(2k)}^k \rightarrow K$ is the inclusion.

Consider the homotopy commutative diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\tilde{f}} & BSC^k \\
 h \downarrow & \searrow \tilde{f} & \uparrow d \\
 V & \xrightarrow{l} & K \\
 g \uparrow & & \uparrow i \\
 U & \xrightarrow{j} & BSC_{(2k)}^k
 \end{array}$$

The maps $d \circ l: V \rightarrow BSC^k$ and $d \circ i \circ j: U \rightarrow BSC^k$ classify k -flat structures $(\xi_1, \nabla_1) \in F_k(U)$, $(\xi_2, \nabla_2) \in F_k(V)$ such that the diagram

$$\begin{array}{ccc}
 \pi^* J_k & \xrightarrow{\pi \nabla_2} & \pi^* V \\
 & \searrow \pi \nabla & \downarrow \\
 & & \Pi^* U \\
 & \searrow & \uparrow \pi(d \circ i \circ j) \\
 & & \pi^* BSC^k
 \end{array}
 \quad (\pi(d \circ l))$$

commutes. (ξ, ∇) is concordant to $h^*(\xi_1, \nabla_1)$ in $F_k(U)$. Hence the diagram

$$\begin{array}{ccc} & \Omega^* M & \\ J_k \nearrow & \uparrow h^* & \\ & \Omega^* V & \end{array}$$

commutes up to chain homotopy, and

$$\begin{array}{ccccc} \pi^* J_k & \xrightarrow{\pi \nabla} & \pi^* M & & \\ & \searrow & \uparrow \pi h & \swarrow \pi \tilde{f} & \\ & & \pi^* V & & \\ & \searrow & \uparrow \pi(dol) & & \\ & & \pi^* BSC^k & & \end{array}$$

commutes precisely.

The complex case works just like the real case with the simplification that BU^k and BC^k are simply connected.

Let $BU_{(2k)}^k = 2k$ -skeleton of BU^k . As before, the inclusion $\tilde{j}: BU_{(2k)}^k \rightarrow BU^k$ lifts

$$\begin{array}{ccc} & BC_C^k & \\ & \downarrow & \\ BH_{(2k)}^k & \xrightarrow{\tilde{j}} & BH^k \end{array}$$

by 3.9(ii), and an argument identical to the one just used proves $\Pi^* I_k$ injects into $\Pi^* BC_C^k$.

COROLLARY 4.9. *There are injections $\Pi^i J_k \rightarrow \Pi^i FC^k$ and $\Pi^i J_k \rightarrow \Pi^i FC_C^k$ in dimensions $i > 2k$.*

PROOF. There is a long exact sequence in homotopy

$$\dots \Pi^i BO^k \xrightarrow{\rho^*} \Pi^i BC^k \xrightarrow{l^*} \Pi^i FC^k \xrightarrow{\delta} \Pi^{i+1} BO^k \rightarrow \dots$$

According to Theorem 4.4, $\Pi^i BO^k = 0$ if $i > 2k$. Therefore, $l^*: \Pi^1 BC_k \rightarrow \Pi^1 FC_k$ is an isomorphism in these dimensions.

The complex case is similar.

We will translate Corollary 4.9 into an equivalent statement about the cohomology of FC^k and FC_C^k .

THEOREM 4.10. *There are injections in cohomology $H^* V_k \rightarrow H^* FC_k$ and $H^* W_k \rightarrow H^* FC_C^k$.*

PROOF. Let \mathcal{H} denote the dual rational Hurewicz map, and $\overline{\mathcal{H}}$, the algebraic dual Hurewicz map [4].

According to [6], given any foliated manifold (M, \mathcal{F}) with trivial normal bundle and adapted connection $\nabla_{\mathcal{F}}$, there is a commutative diagram

$$\begin{array}{ccc} \Pi^* J_k & \xrightarrow{\Pi \nabla} & \Pi^* \mathcal{U} \\ \downarrow \Pi^i & & \uparrow \mathcal{H} \\ \Pi^* W_k & & \\ \uparrow \overline{\mathcal{H}} & & \\ H^* W_k & \xrightarrow{H V_k} & H^* \mathcal{U} \end{array}$$

where $b: \Pi^*W_k \rightarrow \Pi^*I_k$ is a map of graded vector spaces such that $b \circ \overline{\mathcal{H}}: H^*W_k \rightarrow \Pi^*I_k$ is injective and $\Pi i \circ b = \text{id}$. The proof of this theorem extends verbatim to the case of k -flat structures and the diagram restricts as follows:

$$\begin{array}{ccc} \nearrow \Pi^*J_k & \rightarrow & \Pi^*\mathcal{U} \\ \downarrow \Pi i & & \\ \searrow \Pi^*V_k & \uparrow \mathcal{H} & \\ \mathcal{H} \uparrow & & \\ H^*V_k & \xrightarrow{H\nabla_*} & \mathcal{H}^*\mathcal{U} \end{array}$$

By Corollary 4.9, we have an injection $\Pi^i: J_k \rightarrow \Pi^i FC^k$ in dimensions $i > 2k$. Choose a basis of generators $\alpha_1, \dots, \alpha_n$ for the cohomology H^*V_k , and a map $f_i: S^{\gamma_i} \rightarrow FC^k$ detecting the image of α_i under the map

$$H^*V_k \rightarrow \Pi^*V_k \rightarrow \Pi^*J_k \rightarrow \Pi^*FC^k.$$

Let $S = \bigvee_{i=1} S^{\gamma_i}$, $f = \bigvee_{i=1} f^{\gamma_i}: S \rightarrow FC^k$ and let $h: \mathcal{U} \rightarrow S$ be a homotopy equivalence where \mathcal{U} is an open manifold.

$f \circ h: \mathcal{U} \rightarrow FC_k$ classifies a trivial k -flat structure (ξ, ∇) on \mathcal{U} , and in the diagrams above, $\Pi \nabla_k \circ b \circ \overline{\mathcal{H}}$ is injective by construction. Thus, $\mathcal{H} \circ H\nabla_*$ must also be 1-1, and the cohomology H^*V_k injects into the cohomology of \mathcal{U} . The universal properties of FC_k now imply that H^*V_k injects into H^*FC^k .

The complex case is similar.

5. The results in the previous section may be summarized as follows:

1. The exotic characteristic classes H^*W_k are contained in the cohomology of the classifying space FC_C^k for trivialized complex k -flat structures.
2. The subset H^*V_k of H^*W_k of exotic classes is contained in the cohomology of the classifying spaces for FC^k for trivialized real k -flat structures. (It is not known whether all of H^*W_k is contained in H^*FC^k .)

This means in particular that for each $\alpha \in H^*W_k$, there is a manifold M and a k -flat connection ∇ on $M \times R^k$ such that $\nabla_*(\alpha) \neq 0$ in H^*M . We outline below some open questions and methods for generating examples.

Foliations. Recall from the Introduction and §1 that the adapted connection ∇ on the normal bundle to a foliation is k -flat. Hence, the universal k -flat structure induces maps $O_1: B\Gamma_k \rightarrow BC^k$ and $O_2: F\Gamma_k \rightarrow FC^k$. We would like to know the image of $O_2: H^*FC^k \rightarrow H^*F\Gamma_k$.

Discrete groups I. Let $R^k \rightarrow E \xrightarrow{\pi} M$ be a vector bundle whose structure group can be reduced to a discrete subgroup $G \leq Gl(R, k)$. Then E carries a codimension k foliation of transverse to the fibres of π . The adapted connection Π to the normal bundle of the foliation induces a map $\nabla: E \rightarrow BC^k$. Hence precomposition with the zero section s to Π induces a map $\nabla \circ s: M \rightarrow BC^k$. Since this construction is natural, the universality of BG provides maps $\phi_1: BG \rightarrow BC^k$ and $\phi_2: FG \rightarrow FC^k$. As above, we ask: what is the image $\phi_2^*(W_k) \cap H^*FC^k$?

Discrete groups II. We describe a second way to generate k -flat structures from discrete groups.

Let M be a smooth compact k -manifold, $C^\infty(M)$ = smooth functions on M with the discrete topology, $G = Gl_n(C^\infty(M))$. G is the group of trivializations of the bundle $R^n \rightarrow M \times R^n \rightarrow M$. Therefore, a G -structure on a manifold W (i.e. a map $W \rightarrow BC$) determines a principal G -bundle $\xi: G \rightarrow X \xrightarrow{\pi} W$.

Let (U_α, ϕ_α) be a trivialization of ξ , and define $E = \bigcup_\alpha U_\alpha \times M \times R^n / \sim$ where $(w, m, v) \in U_\alpha \times M \times R^n \sim (w, m, \phi_\beta^{-1} \circ \phi_\alpha(w)(m)(v)) \in U_\beta \times M \times R^n$ whenever $w \in U_\alpha \cap U_\beta$.

E is an R^n vector bundle over $W \times M$ whose restriction to the W factor is flat (since G has the discrete topology). Hence, E is k -flat.

The above construction determines a map $\phi: BG \rightarrow BC^k$, and it is a fundamental problem to determine how much of the cohomology of BC^k can be detected by such cycles. See [18].

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